A realistic modelling of the dynamics of equity volatility

Hervé Andrès, Alexandre Boumezoued

Real-world (RW) economic scenarios are scenarios that appear credible with respect to what happened in the past. Simple modelling approaches which are popular in practice often fail to fit the historical distributions very well. It is the case for popular models of equity stocks of indices that assume constant volatility while historical time series clearly demonstrate that volatilities vary significantly over time.

In this paper, we describe a recent modelling approach of the volatility, based on the fractional Brownian motion, which is highly consistent with historical data.

Real-world economic scenarios have become a key tool for insurance companies for applications requiring deriving realistic distributions of the balance sheet. These applications cover asset and liability management (ALM) studies, computing the Solvency Capital Requirement (SCR) within an Internal model, or pricing assets or liabilities including a risk premium. Unlike risk-neutral (RN) economic scenarios, RW ones should be realistic in view of the historical data and/or management expectations about future outcomes (e.g., a further unlimited falling of interest rates doesn't appear to be likely even if it is "suggested" by historical data). However, in practice, the majority of the features of RW scenarios are calibrated to the history because in most cases history is the most objective predictor for the future. There are two possible approaches for measuring how consistent a model is with respect to historical data:

1. To evaluate the ability of the model to replicate some statistical properties of historical data (for example, the histogram of increments).
2. To evaluate the ability of the model to satisfy some empirical properties, often called stylised facts. See, for example, Cont (2001).[1]

These two approaches are complementary as the second one allows us to capture path-wise properties (for example volatility clustering) that the first approach doesn't capture. Thus, we propose to use these approaches as measures of the ability of models to replicate historical data.

In this paper, we focus on equity models. Within the insurance industry, the most widespread model for generating equity paths is the celebrated Black-Scholes model, which models the spot price of a stock or index under the RW probability according to the following dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ is the instantaneous return of the stock (also called drift), σ is the volatility, and $(W_t)_{t\geq 0}$ is a standard Brownian motion

Sometimes, the drift and/or the volatility are assumed to be deterministic functions of the time. The main advantages of this model are its simplicity and the fact that it allows us to preserve the consistency between the RW and RN frameworks.

One of the underlying assumptions of this model is the normality and independence of the log-increments $\log \frac{S_t}{S_{t-1}}$ at any timescale. However, this assumption doesn't hold true in practice, as shown in Figures 1 and 2. In Figure 1, one can indeed observe that the empirical distribution of historical log-increments is skewed, i.e., not symmetric around the mean, unlike the normal distribution. In particular, the empirical distribution exhibits a fatter left tail than the normal distribution.

This is more visible in Figure 2 where it appears that the smallest quantiles of the normal distribution are much lower than those of the empirical distribution. This is particularly undesirable if one wants to compute the SCR using this model because it will lead to an underestimation of the 0.5% worst one-year deviation of the considered equity portfolio.

FIGURE 1: HISTOGRAM OF MONTHLY LOG-INCREMENTS (IN BLUE) OF THE S&P 500 FROM JANUARY 1980 TO DECEMBER 2020, CALIBRATED NORMAL DENSITY IN RED

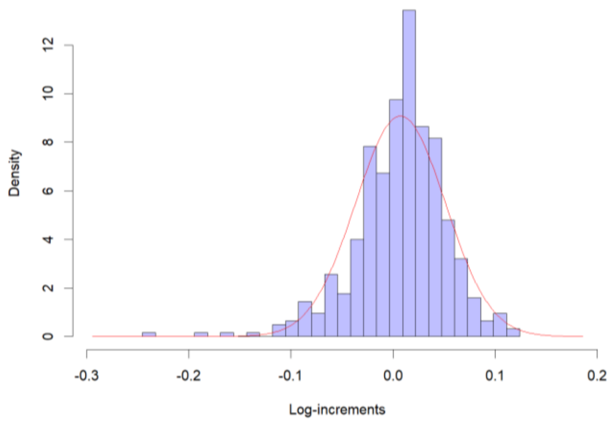
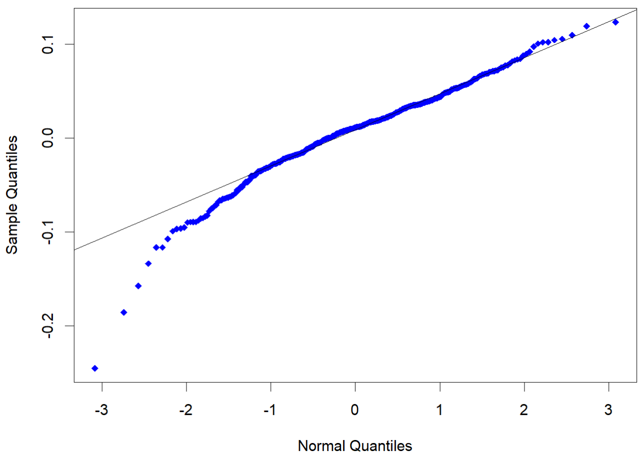


FIGURE 2: QQPLOT COMPARING THE EMPIRICAL QUANTILES OF THE S&P 500 MONTHLY LOG-INCREMENTS TO THE NORMAL QUANTILES



A natural way to improve this modelling is to assume that the volatility is not deterministic anymore but stochastic. Such extension is not new and the first successful attempt to construct a continuous-time model with stochastic volatility is due to Heston (1993).[2] In the original paper, this model has been constructed for RN modelling, but similar dynamics could be also considered in a RW framework, where the parameters of the Cox-Ingersoll-Ross stochastic volatility dynamics are derived from the price time series. An alternative modelling approach is to use a Generalised AutoRegressive Conditional Heteroscedasticity (GARCH) process. Although it is not strictly speaking a continuous-time stochastic volatility model, it allows us to achieve a good fit to financial time series. Moreover, it allows for volatility clustering, which is a widely accepted stylised fact.

In this paper, we present and revisit an analysis of reconstructed historical spot volatility conducted by Gatheral et al. (2018) in their seminal paper "Volatility Is Rough".[4] Their main conclusion is that the behaviour of historical log-volatility is very close to the behaviour of a fractional Brownian motion with Hurst parameter of order 0.1, which is less regular ("rougher") than the standard Brownian motion. This analysis suggests that RW equity models should use rough volatility in order to be realistic.

A brief introduction to the fractional Brownian motion

The fractional Brownian motion (fBm), is a generalisation of the Brownian motion. It depends on parameter $H \in (0,1)$, called the Hurst parameter, whose value parametrises several properties of the fBm. Furthermore, its increments are not necessarily independent.

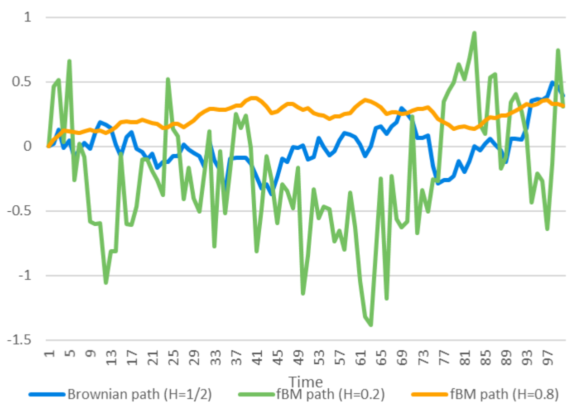
Formally, the fBm $(W_t^H)_{t\in\mathbb{R}}$ is a centered self-similar Gaussian process with stationary increments satisfying the following scaling property:

$$\forall t \in \mathbb{R}, \Delta \geq 0, q > 0 : \mathbb{E}\left[\left|W_{t+\Delta}^H - W_t^H\right|^q\right] = K_q \Delta^{qH} \quad (1)$$

with $K_q = \mathbb{E}[|G|^q]$ and $G \sim \mathcal{N}(0,1)$. In particular, the increments $W_{t+\Delta}^H - W_t^H$ of a fBm are normally distributed with mean 0 and variance Δ^{2H}.

Moreover, when $H = \frac{1}{2}$ the fBm reduces to the standard Brownian motion (and independence between increments is recovered in that case).

FIGURE 3: FBM SAMPLE PATHS FOR DIFFERENT VALUES OF THE HURST PARAMETER



The Hurst parameter H allows us to control the regularity (in the sense of Hölder) of the sample paths of W^H: when H is closer to one, the sample paths become more regular than those of the standard Brownian motion and when H is closer to zero, the sample paths become less regular.[5] A comparison of the regularity of sample paths for different values of Hurst parameter is shown in Figure 3.

Again, one of the main differences of the fBm from the standard Brownian motion is that the increments are not independent (except if $H = \frac{1}{2}$): they keep memory of what happened in the past. More precisely, for $H \in \left(0, \frac{1}{2}\right)$, the fBm has the property of counterpersistence (related to mean reversion): if it was increasing in the past, it is more likely to decrease in the future and vice versa. In contrast, for $H \in \left(\frac{1}{2}, 1\right)$, the fBm is persistent: it is more likely to keep trend than to break it. Moreover, in the case $H \in \left(\frac{1}{2}, 1\right)$, the fBm is said to have long memory because the autocorrelation function decays very slowly.

Historical volatility analysis

For a long time, the long memory property of the volatility process has been considered as a stylised fact due in particular to the papers from Ding et al. (1993),[6] Andersen and Bollerslev (1997)[7] and Andersen et al. (2001).[8] This has motivated (among other reasons) Comte and Renault (1998)[9] to model log-volatility using a fBm with a Hurst parameter $H > 1/2$ to ensure long memory. However, the long memory property and the choice of $H > \frac{1}{2}$ has been recently challenged by Gatheral et al. (2018) as they estimated the Hurst parameter on reconstructed historical volatility time series and obtained values between 0.08 and 0.2, which indicates that the volatility doesn't have long memory. In the following, we briefly describe and reproduce the analysis they performed, and we discuss the implications for RW modelling at relevant simulation timescales within the insurance industry.

Let us introduce

$$m(q,\Delta) = \frac{1}{N}\sum_{k=1}^{N} \left|\log\sigma_{k\Delta} - \log\sigma_{(k-1)\Delta}\right|^{q}$$

where $\Delta > 0$, $q \geq 0$, $N = \lfloor T/\Delta \rfloor$ and $\sigma_0, \sigma_\Delta, \ldots, \sigma_{N\Delta}$ are discrete observations of the volatility process of an equity index.

Assuming stationary log-increments and that a law of large numbers can be applied, $m(q,\Delta)$ is an unbiased estimator of $\mathbb{E}[|\log\sigma_\Delta - \log\sigma_0|^q]$. If the log-volatility behaves as a fBm, we expect that $\log m(q,\Delta)$ is essentially an affine function of $\log\Delta$ with slope qH due to equation (1).

Using daily realised volatility estimates[10] from the Oxford-Man Institute of Quantitative Finance Realized Library[11] (from 2000 to 2021) as a proxy to the spot volatility (which is unobserved), we can plot $\log m(q,\Delta)$ as a function of $\log\Delta$ for Δ ranging from one day to 360 days. The obtained graphs for the S&P 500 and FTSE 100 historical volatility are given in Figures 4 and 5.

We observe that for each value of q, $\log m(q,\Delta)$ looks like an affine function of $\log\Delta$ especially for low and medium values of Δ, which is not surprising because, for low and medium values of Δ, $m(q,\Delta)$ is estimated on more data points. If we restrict Δ between 20 and 40 days, that is around one month, which is generally the time step considered in RW modelling in insurance, and then we obtain Figures 6 and 7. Moreover, if we plot the slope ξ_q of each of the affine functions as a function of q, we obtain a linear function (intercept is very close to zero) as shown by Figure 8.

FIGURE 4: $\log(m(q,\Delta))$ AS A FUNCTION OF $\log(\Delta)$ FOR THE S&P 500 FOR $\Delta = 1, \ldots, 360$

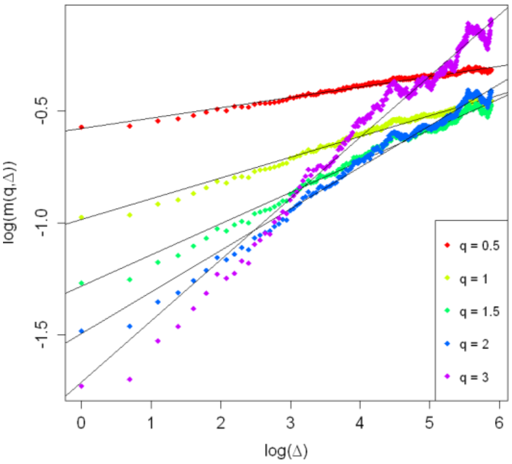


FIGURE 5: $\log(m(q,\Delta))$ AS A FUNCTION OF $\log(\Delta)$ FOR THE FTSE 100 FOR $\Delta = 1, \ldots, 360$

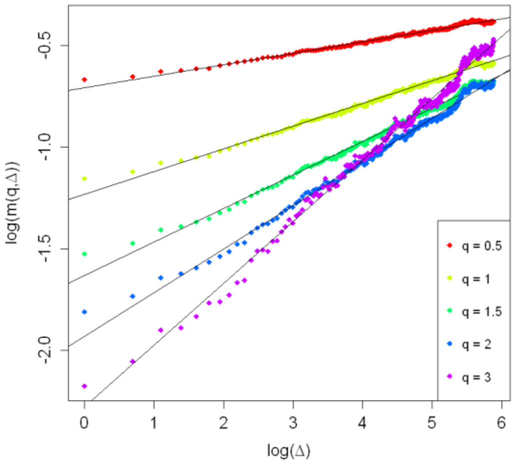


FIGURE 6: $\log(m(q,\Delta))$ AS A FUNCTION OF $\log(\Delta)$ FOR THE S&P 500 FOR $\Delta = 20, \ldots, 40$

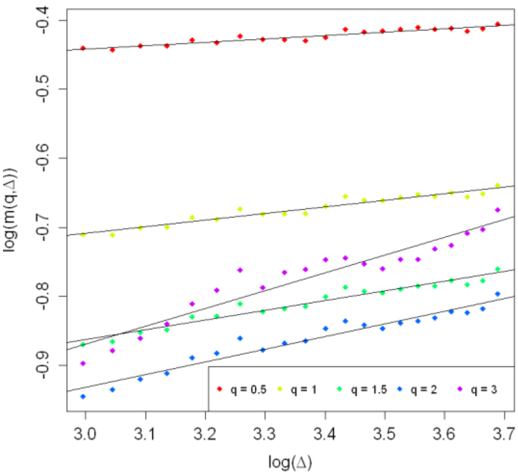
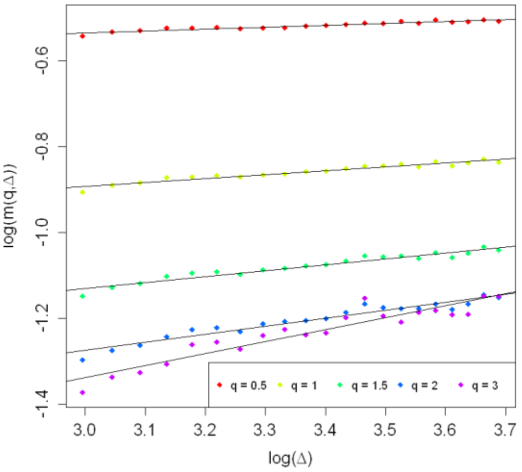


FIGURE 7: $\log(m(q,\Delta))$ AS A FUNCTION OF $\log(\Delta)$ FOR THE FTSE 100 FOR $\Delta = 20, \dots, 40$



These results show that:

$$\log m(q,\Delta) \approx \xi_q \times \log \Delta + b_q$$

and

$$\xi_q \approx qH.$$

The Hurst parameter H is estimated as 0.09 for the S&P 500 and 0.10 for the FTSE 100. This behaviour of $m(q,\Delta)$ is fully consistent with the scaling property of the fBm given in equation (1). Moreover, if we look at the histogram of the monthly log-increments of the volatility, we obtain a distribution that is very close to a Gaussian distribution, as shown by Figures 9 and 10. It is interesting to observe that rescaling the standard deviation obtained from the normal fit of one-day log-increments by Δ^H (that is, the theoretical standard deviation of a fBm) produces a curve that is very close to a fitted normal density on monthly log-increments (by log-likelihood maximisation).

FIGURE 8: SLOPE ξ_q OF AS A FUNCTION OF q FOR S&P 500 (LEFT) AND FTSE 100 (RIGHT)

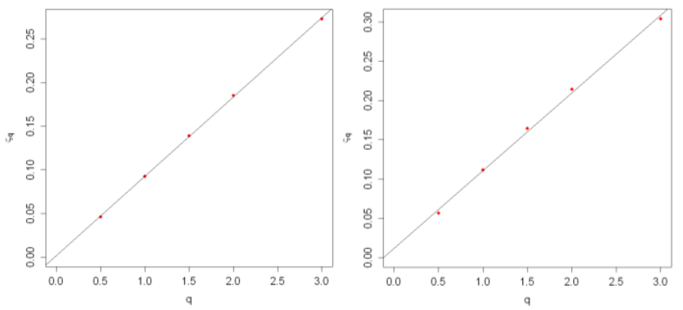
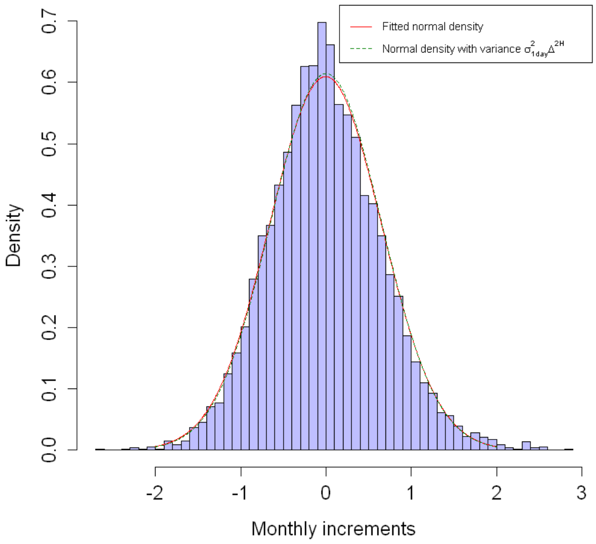


FIGURE 9: HISTOGRAM OF MONTHLY LOG-INCREMENTS FOR THE S&P 500



To summarise, this analysis shows that, empirically, the volatility log-increments behave as a fBm with Hurst parameter near 0.1. While this may seem to contradict the papers mentioned previously that conclude long memory of the volatility (and therefore $H > \frac{1}{2}$ for a fBm), Gatheral et al. show that there is no contradiction. For this purpose, they apply the statistical procedure used by Andersen et al. (2001) to detect long memory on simulated fBm paths with H near 0.1. The procedure identifies long memory, which proves that the procedure is fragile and can lead to inaccurate conclusions of long memory.

Therefore this empirical study suggests the following model:

$$\log \sigma_{t+\Delta} - \log \sigma_t = \nu\left(W^H_{t+\Delta} - W^H_t\right) \qquad (2)$$

which depends on only two parameters ν and H.

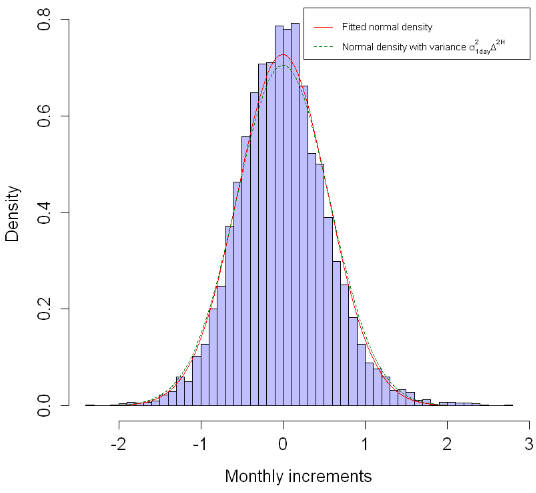
However, this model is not stationary, which implies that the volatility could become arbitrarily large at very large times. This is not desirable because we observe empirically that periods of high volatility are generally limited in time and followed by periods of low volatility (see top graph of Figure 11 below for instance). Following Gatheral et al., we modify model (2) by adding a mean-reverting term to ensure stationarity:

$$dX_t = \nu dW^H_t - \alpha(X_t - m)dt \qquad (3)$$

where $X_t = \log \sigma_t$, α is the mean-reverting speed and m is the mean-reverting level.

Note that model (3) is a fractional version of the Ornstein-Uhlenbeck dynamics. Gatheral et al. suggest taking a small value of α so that the log-volatility behaves locally as a fBm.

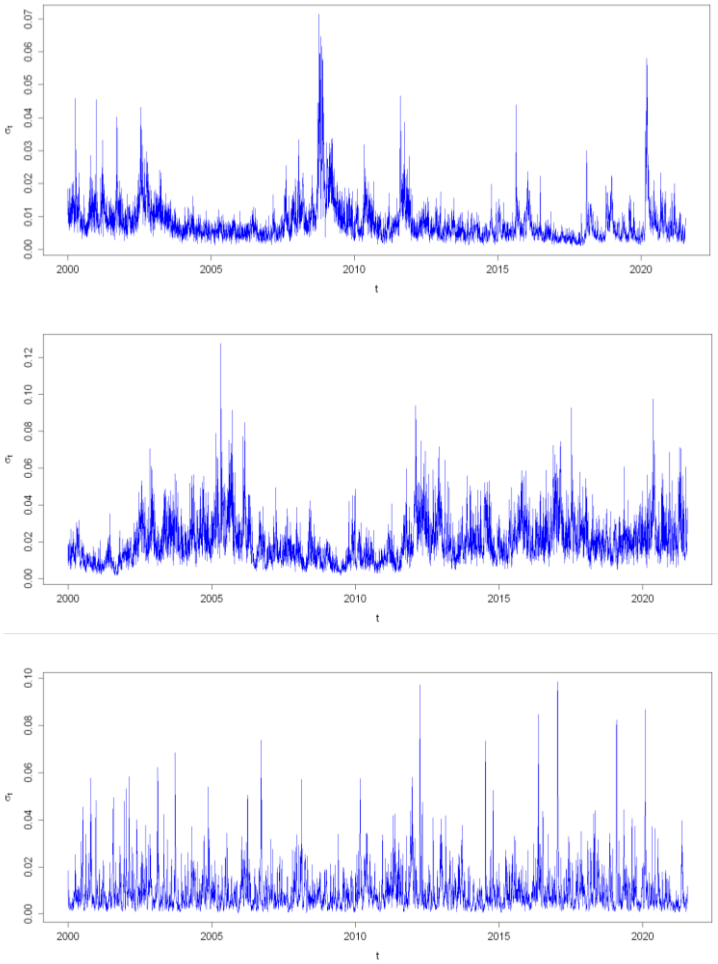
FIGURE 10: HISTOGRAM OF MONTHLY LOG-INCREMENTS FOR THE S&P 500



In order to show that the model (3) is more realistic than a model based on the standard Brownian motion, we compare in Figure 11 the historical (reconstructed) daily volatility path of the S&P 500 from the Oxford-Man Institute to a path generated by model (3) and a path generated by the same model but with a standard Brownian motion instead of a fBm. The Hurst parameter has been set to the value obtained through the linear regression and the scaling parameter ν has been set to the standard deviation of the one-day volatility log-increments. The mean-reverting speed α has been set to zero for the fractional version of model (3) for simplicity but simulation of paths with a non-zero mean-reversion speed is straightforward for applications that call for it. On the other hand, for the path generated by model (3) with a standard Brownian motion, a non-zero mean-reversion speed has been used in order to avoid explosions of the volatility.

The comparison is only visual, but it clearly appears that the model based on the fractional Brownian motion produces a path that is more like the historical path than the model based on the standard Brownian motion. This correspondence appears better due to the clustering behaviour of the fBm along with the rougher paths obtained. Note, however, that the model is not able to generate too-long periods of low volatility, as we can observe in the historical data (for example between 2003 and 2008). One way to refine the model would be therefore to include regime switches into the model, for instance by using a hidden Markov chain.

FIGURE 11: EVOLUTION OF THE HISTORICAL (RECONSTRUCTED) VOLATILITY OF THE S&P 500 (TOP GRAPH) ALONG WITH A SIMULATED PATH OF MODEL (2) WITH A FRACTIONAL BROWNIAN MOTION (MIDDLE GRAPH) AND A SIMULATED PATH OF MODEL (2) WITH A STANDARD BROWNIAN MOTION (BOTTOM GRAPH)



Conclusion

Although RW modelling has many applications in the insurance industry, some existing models remain quite limited in their ability to replicate historical dynamics and properties. In this paper, we present a new approach proposed by Gatheral et al. (2018) for the modelling of equity volatility that is highly consistent with historical data. More precisely, the proposed model has the following main advantages for RW modelling:

1. It allows us to replicate the statistical distribution of the empirical volatility (Gaussian distribution).
2. It allows us to replicate the regularity of the paths of the empirical volatility (scaling property).
3. It depends on a small number of parameters that can be easily estimated.
4. It seems to be universal—we provided results for two major equity indices but similar results have been obtained for many other equity indices as well as individual equities.[12]
5. It is justified from an economic point of view as El Euch et al. (2018)[13] showed by building a microscopic model for the price that the typical behaviours of market participants at the high-frequency scale generate rough volatility, i.e ., volatility behaving as a fBm with Hurst parameter smaller than $\frac{1}{2}$.Endnotes.

CONTACT

If you have any questions or comments on this paper or any other aspect of Economic Scenarios Generation, please contact your usual Milliman consultant or the email address below:

chess@milliman.com

References

[1] Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. Quantitative finance, 1(2), 223.

[2] Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. The review of financial studies, 6(2), 327-343.

[4] Gatheral, J., Jaisson, T., & Rosenbaum, M. (2018). Volatility is rough. Quantitative finance, 18(6), 933-949.

[5] Mathematically, a fBm with Hurst parameter H is r-Hölder for all $r < H$, i.e.,

$$\sup_{s \neq t} \frac{|W_t^H - W_s^H|}{|t-s|^r} < \infty.$$

[6] Ding, Z., Granger, C. W. & Engle, R. F. (1993). A long memory property of stock market returns and a new model. Journal of empirical finance, 1(1), 83-106.

[7] Andersen, T. G. & Bollerslev, T. (1997). Intraday periodicity and volatility persistence in financial markets. Journal of empirical finance, 4(2-3), 115-158.

[8] Andersen, T. G., Bollerslev, T., Diebold, F. X. & Labys, P. (2001). The distribution of realized exchange rate volatility. Journal of the American statistical association, 96(453), 42-55.

[9] Comte, F. & Renault, E. (1998). Long memory in continuous-time stochastic volatility models. Mathematical Finance, 8(4), 291-323.

[10] These estimates are obtained using intraday high-frequency data.

[11] Oxford-Man Institute of Quantitative Finance (7 January 2022). Realized Library. Retrieved 9 January 2022 from http://realized.oxford-man.ox.ac.uk/data/download.

[12] Bennedsen, M., Lunde, A., & Pakkanen, M. S. Decoupling the Short-and Long-Term Behavior of Stochastic Volatility. Journal of Financial Econometrics. 2021.

[13] El Euch, O., Fukasawa, M., & Rosenbaum, M. (2018). The microstructural foundations of leverage effect and rough volatility. Finance and Stochastics, 22(2), 241-280.